JOURNAL OF APPROXIMATION THEORY 63, 137–143 (1990)

Approximate Units and Monotone Convergence*

LEONEDE DE MICHELE AND DELFINA ROUX

Dipartimento di Matematica, Università di Milano, 20133 Milano, Italy

Communicated by P. L. Butzer

Received August 15, 1988

1. INTRODUCTION

In a very recent paper, L. Colzani [2] studied the problem of monotone approximation of a function by its Fejér means. Of course, the approximation is monotone in $L^2(T)$, but he proved that this is no longer true in $C(T)$. Nevertheless, there is a good control of the oscillations of this approximation.

In this paper we show that it is hopeless to find a monotone approximation in $C(T^N)$ whatever is the approximate unit we can use and we prove that results similar to that for the Fejér means hold for a large class of approximate units.

2. RESULTS

If $N \geqslant 1$, let Z^N be the lattice of integer points of R^N and $T^N = R^N/Z^N$ the N-dimensional torus. Let us denote by B, indifferently, the Lebesgue space $L^p(T^N)$, $1 \leqslant p < +\infty$, or the space of continuous functions $C(T^N)$ and denote their norm by $\| \ \|_B$; for convenience we identify T^N with $[-\frac{1}{2}, \frac{1}{2})^N$.

Let us recall that an approximate unit (or summability kernel) is a family $\{K_\alpha\}_{\alpha \in R^-}$ where

(i) $K_\alpha \in L^1(T^N)$ and $\|K_\alpha\|_1 \leqslant M \ \forall \alpha$;

(ii) $\int_{T^N} K_\alpha(t)\, dt = 1 \ \forall \alpha > 0$;

(iii) for every $\varepsilon > 0$ $\lim_{\alpha \to 0+} \int_{|t| \geqslant \varepsilon,\, t \in T^N} |K_\alpha(t)|\, dt = 0$.

It is well known (see, e.g., [1, p. 31]) that if $\alpha \to 0+$

$$\|K_\alpha * f - f\|_B \to 0$$

* Work partially supported by the Italian M.P.I.

640-63/2-2

0021-9045/90 $3.00

and obviously by the Plancherel theorem if $|\hat{K}_\alpha(j)| \uparrow 1$ $\forall j$, then $\|K_\alpha * f - f\|_2 \downarrow 0$.

Now we show that if $B = C(T^N)$ this is never the case. Indeed for every given $\{K_\alpha\}_{\alpha \in R^-}$ and for every $\alpha_1 > \alpha_2 > 0$ there exists a function f such that

$$\|K_{\alpha_2} * f - f\|_\infty > \|K_{\alpha_1} * f - f\|_\infty .$$

This result follows from

THEOREM 1. *Let* $K_1, K_2 \in L^1(T^N)$, $K_1 \neq K_2$. *Then there exist* $g_1, g_2 \in C(T^N)$ *with* $\|g_1\|_\infty = \|g_2\|_\infty = 1$ *such that*

$$\|(\delta_0 - K_1) * g_1\|_\infty > \|(\delta_0 - K_2) * g_1\|_\infty \tag{2.1}$$

$$\|(\delta_0 - K_1) * g_2\|_\infty < \|(\delta_0 - K_2) * g_2\|_\infty , \tag{2.1$'$}$$

where δ_0 *is the unit mass in the origin.*

Making some more assumptions about $\{K_\alpha\}_{\alpha \in R^-}$ and generalizing the methods of [2], we can give more refined results, which hold in particular for classical kernels.

THEOREM 2. *Let us suppose* $K_\alpha \in L^2(T^N)$ *and* $\hat{K}_\alpha(j) \neq 1$ *for every* $j \neq 0$ *and for every* α. *Then for every* $\beta > 0$ *and* $\varepsilon > 0$ *there exist* α, $0 < \alpha < \beta$, *and* $f \in C(T^N)$ *such that*

$$\|f - K_\alpha * f\|_\infty > (2 - \varepsilon) \|f - K_\beta * f\|_\infty . \tag{2.2}$$

THEOREM 3. *With the same hypotheses as in Theorem* 2, *for every* $\beta > 0$ *let us set* $A_\beta = \sup_{\alpha \leqslant \beta} \|K_\alpha\|_1$. *Then there exist* $\varphi_\beta, \psi_\beta \colon R^+ \to R^+$ *with* $\lim_{\alpha \to 0} \varphi_\beta(\alpha) = 0$, $\psi_\beta(\alpha) \geqslant 1$, $\limsup_{\alpha \to 0} \psi_\beta(\alpha) \leqslant 1 + A_\beta$, *such that if* $f \in B$ *and* $\alpha < \beta$

$$\|f - K_\alpha * f\|_B \leqslant \psi_\beta(\alpha)^{2|1 - 1/p|} \{1 + A_\beta + \varphi_\beta(\alpha)\}^{|2/p - 1|} \|f - K_\beta * f\|_B . \tag{2.3}$$

3. PROOFS

Proof of Theorem 1. Let $\sigma > 0$ be such that if $E \in T^N$, $|E| < \sigma$ ($|E|$ is the Lebesgue measure of E),

$$\int_E |K_i(t)|\, dt < \tfrac{1}{10}, \qquad i = 1, 2. \tag{3.1}$$

Since $\int_{T^N} |K_1(t) - K_2(t)| \, dt \neq 0$, there exist a sphere $S = S(t_0, \rho)$ with radius $\rho < \|t_0\|/2$ and $|S| < \sigma/2^N$ and a function $\varphi \in C(T^N)$ with support in S such that

$$\|\varphi\|_\infty = \frac{1}{2}, \qquad \int_S (K_1(t) - K_2(t)) \, \varphi(t) \, dt = a, \tag{3.2}$$

where a is a real positive number.

If

$$\gamma = \int_S K_2(t) \, \varphi(t) \, dt$$

we have $|\gamma| < \frac{1}{20}$ and

$$\left| 1 + \int_S K_1(t) \, \varphi(t) \, dt \right| = |1 + a + \gamma| = |1 + \gamma| + b$$

with $b > 0$.

Let us take r, $0 < r < \frac{1}{2}\min(\sigma, \|t_0\| - 2\rho)$ such that for every t, $\|t\| \leqslant r$,

$$|K_i * \varphi(t) - K_i * \varphi(0)| < \frac{b}{4}, \qquad i = 1, 2. \tag{3.3}$$

Let $\psi \in C(T^N)$ be a non-negative function with support in $S(0, r)$ such that $\psi(0) = 1$, $\|\psi\|_\infty = 1$, and

$$|K_i * \psi(t)| < \frac{b}{4} \qquad \forall t \in T^N, \qquad i = 1, 2. \tag{3.4}$$

Let us set now

$$g_i(t) = \psi(t) + (-)^i \, \varphi(t), \qquad i = 1, 2.$$

By construction, we obviously have $\|g_1\|_\infty = 1$ and

$$\begin{aligned} |(\delta_0 - K_1) * g_1(0)| &= \left| 1 + \int_S K_1(t) \, \varphi(t) \, dt - K_1 * \psi(0) \right| \\ &> |1 + \gamma| + \frac{3}{4} b. \end{aligned} \tag{3.5}$$

Moreover, if $\|t\| > r$, by (3.1) we have

$$|(\delta_0 - K_2) * g_1(t)| < \tfrac{1}{2} + \tfrac{1}{10} + \tfrac{1}{20} = \tfrac{13}{20}. \tag{3.6}$$

In the case $\|t\| \leqslant r$ we have

$$|(\delta_0 - K_2) * g_1(t)| = |\psi(t) - K_2 * \psi(t) + K_2 * \varphi(t)|$$

and, by (3.3) and (3.4),

$$\begin{aligned} |(\delta_0 - K_2) * g_1(t)| &\leqslant |\psi(t) + K_2 * \varphi(0)| + \frac{b}{2} \\ &= |\psi(t) + \gamma| + \frac{b}{2}. \end{aligned}$$

Because $0 \leqslant \psi(t) \leqslant 1$ and $|\gamma| < \frac{1}{20}$ we have, for $\|t\| \leqslant r$,

$$|(\delta_0 - K_2) * g_1(t)| < |1 + \gamma| + \frac{b}{2}. \tag{3.7}$$

Then by (3.5), (3.6), and (3.7) it follows that g_1 verifies (2.1).
In the same way it can be proved that g_2 verifies (2.1′).

For the sequel we need the following

LEMMA. *Given the same hypotheses as in Theorem* 2,

$$(f - K_\alpha * f)^\wedge = \Phi_{\alpha,\beta}(f - K_\beta * f)^\wedge, \tag{3.8}$$

where $\Phi_{\alpha,\beta}$ *is the Fourier–Stieltjes transform of a Borel measure* $\nu_{\alpha,\beta}$ *of the form*

$$\nu_{\alpha,\beta} = \delta_0 - K_\alpha + \mu_{\alpha,\beta} \tag{3.9}$$

with $\|\mu_{\alpha,\beta}\|_M \to 0$ *if* $\alpha \to 0+$ *for every* $\beta > 0$, *where* $\|\ \|_M$ *is the usual total variation of the measure.*

Proof. We have

$$\begin{aligned} (f - K_\alpha * f)^\wedge &= \frac{(f - K_\alpha * f)^\wedge}{(f - K_\beta * f)^\wedge} (f - K_\beta * f)^\wedge \\ &= \Phi_{\alpha,\beta} \cdot (f - K_\beta * f)^\wedge, \end{aligned}$$

where for every $j \neq 0$

$$\Phi_{\alpha,\beta}(j) = 1 - \hat{K}_\alpha(j) + \hat{K}_\beta(j) \cdot \frac{1 - \hat{K}_\alpha(j)}{1 - \hat{K}_\beta(j)}.$$

Because $(1-\hat{K}_\alpha(j))/(1-\hat{K}_\beta(j))$ is bounded, $\Phi_{\alpha,\beta}=\hat{\nu}_{\alpha,\beta}$ and the measure $\mu_{\alpha,\beta}$ in (3.9) is such that

$$\hat{\mu}_{\alpha,\beta}=\hat{K}_\beta\cdot\frac{1-\hat{K}_\alpha}{1-\hat{K}_\beta}.$$

Then $\|\mu_{\alpha,\beta}\|_2\to 0$ if $\alpha\to 0+$ for every $\beta>0$. Since $\|\mu_{\alpha,\beta}\|_1\leqslant\|\mu_{\alpha,\beta}\|_2$ the Lemma is proved.

Proof of Theorem 2. Let $\varepsilon>0$. By (3.9) and $\hat{K}_\alpha(0)=1$ we have for every $\beta>0$

$$\lim_{\alpha\to 0}\|\nu_{\alpha,\beta}\|_M>2-\frac{\varepsilon}{2}. \tag{3.10}$$

Then for α small enough there exists a continuous function g with $\|g\|_\infty=1$ such that

$$\nu_{\alpha,\beta}*g(0)\geqslant 2-\varepsilon$$

and g can be chosen with $\hat{g}(0)=0$ because of (iii).

Since

$$\frac{1}{1-\hat{K}_\beta(j)}=\left(1+\frac{\hat{K}_\beta(j)}{1-\hat{K}_\beta(j)}\right),\qquad j\neq 0,$$

there exists a Borel measure μ_β such that $\hat{\mu}_\beta=1/(1-\hat{K}_\beta)$ for $j\neq 0$. This implies that there exists a continuous function f such that

$$\hat{f}=\frac{\hat{g}}{1-\hat{K}_\beta}\qquad\text{if}\quad j\neq 0.$$

For such a function f we have

$$f-K_\beta*f=g$$

and (3.8), (3.10) give

$$\begin{aligned}\|f-K_\alpha*f\|_\infty&=\|\nu_{\alpha,\beta}*g\|_\infty\geqslant(2-\varepsilon)\,\|g\|_\infty\\&\geqslant(2-\varepsilon)\,\|f-K_\beta*f\|_\infty.\end{aligned}$$

Proof of Theorem 3. By (3.9) we have for every $\beta>0$

$$\|\nu_{\alpha,\beta}\|_M\leqslant 1+A_\beta+\varphi_\beta(\alpha)$$

with $\varphi_\beta(\alpha)\to 0$ if $\alpha\to 0+$.

Then (2.3) holds with $\psi_\beta(\alpha)=1$ in the case $B=L^1(T^N)$ or $B=C(T^N)$. If $B=L^2(T^N)$ then since

$$\|f-K_\alpha*f\|_2^2=\sum|\hat{f}(j)|^2\cdot\left|\frac{1-\hat{K}_\alpha(j)}{1-\hat{K}_\beta(j)}\right|^2\cdot|1-\hat{K}_\beta(j)|^2$$

we have

$$\begin{aligned}\|f-K_\alpha*f\|_2&\leqslant\sup_{j\neq 0}\left|\frac{1-\hat{K}_\alpha(j)}{1-\hat{K}_\beta(j)}\right|\cdot\|f-K_\beta*f\|_2\\&=\psi_\beta(\alpha)\cdot\|f-K_\beta*f\|_2.\end{aligned}$$

Obviously, $\psi_\beta(\alpha)$ satisfies the hypotheses in the statement.
By interpolation we get (2.3).

4. Remarks

1. Theorem 1 is trivial if there exist j_1, j_2 such that

$$|1-\hat{K}_1(j_1)|>|1-\hat{K}_2(j_1)|,\qquad |1-\hat{K}_1(j_2)|<|1-\hat{K}_2(j_2)|.$$

2. The proofs of Theorem 2 and the lemma show that the hypothesis $K_\alpha\in L^2(T^N)$ $(\alpha\in R^+)$ is only used to prove that for every $\beta>0$ the function $1/(1-\hat{K}_\beta)$ is a Fourier–Stieltjes transform of a Borel measure, for $j\neq 0$. Then Theorems 2 and 3 hold in many other situations.

3. Usually, $\|K_\alpha\|_1=1$ for every α. In this case $A_\beta=1$. If moreover $\hat{K}_\alpha(j)\uparrow 1$ if $\alpha\to 0+$ for every j, then Theorem 3 has a more appealing form. Indeed (2.3) becomes

$$\|f-K_\alpha*f\|_B\leqslant\{2+\varphi_\beta(\alpha)\}^{|2/p-1|}\,\|f-K_\beta*f\|_B.$$

4. We have already observed that Theorems 2 and 3 hold for the classical kernels: Fejér, Poisson, Gauss. Moreover it is possible to apply these theorems to other cases, such as the kernels K_σ studied in [3], where $\hat{K}_\sigma(n)=1/(1+\sigma P(n))$, and P is a suitable homogeneous polynomial of degree k, when $k>N/2$, that is the more important case for the applications.

5. It is worth mentioning that for the Gauss–Weierstrass kernel in R there is monotone convergence for the class of convex functions [1, p. 154]. This suggests that Theorem 2 may be no longer true for particular kernels if we restrict ourselves to suitable subclasses of $C(T^N)$.

REFERENCES

1. P. L. BUTZER AND R. J. NESSEL, "Fourier Analysis and Approximation I," Birkhäuser, Basel/Stuttgart, 1971.
2. L. COLZANI, Is the approximation of a function by its Fejér means monotone?, *J. Approximation Theory* **56** (1989), 152–154.
3. L. DE MICHELE, L. GOTUSSO, AND D. ROUX, Some theoretical results suggested by numerical experience, *in* "Approximation Theory and Applications," pp. 14–29, Research Notes in Mathematics, Vol. 133, Pitman, Boston, 1985.